Entanglement Entropy (Conical Entropy) in String Theory

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JHEP05(2015)106.

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September 17, 2015

@ ITP, PKU

Outline

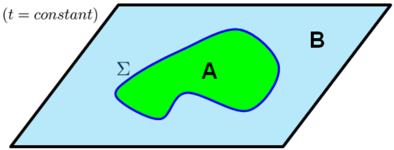
1. Motivation and Background of Entanglement entropy (EE).
2. EE for free fields with spins.
3. (Twisted) Conical Entropy in String theory.
4. Summary and comments.

Basics of EE

- EE is a useful measure of the degrees of freedom in quantum many body systems.
  1. Using EE to detect the central charge (the coefficient of logarithmic divergent term in Odd dimesion)[C. Holzhey, F. Larsen and F. Wilczek, 94][P. Calabrese and J. L. Cardy, 04][S. Ryu and T. Takayanagi,06][...].
  2. Detecting the topological degrees of freedom of topological field theories (finite piece of EE)[A. Kitaev and J. Preskill,05][M. Levin and X.G.Wen,05].
  3. Measuring the degrees of freedom of local operators (Quantum dimension).[M. Nozaki, T. Numasawa and T. Takayanagi,14][S. He, T. Numasawa, T. Takayanagi and K. Watanabe,14][P. Capta, M. Nozaki and T. Takayanagi, 14][M. Nozaki,14][Wu-Zhong Guo, S. He,15].
- ...

Basics of EE

- General diagnostic: divide quantum system into two parts (A and B) and use entropy as measure of correlations between subsystems

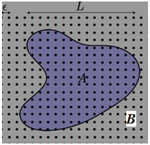


- Integrate out degrees of freedom in outside region (B). Remaining dof are described by a density matrix ρ_A.

$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A \tag{1}$$

Main motivation

- EE has following UV structure in various field theory:



In even dimensions

$$S_A = C_{d-2}\frac{L^{d-2}}{\epsilon^{d-2}} + C_{d-4}\frac{L^{d-4}}{\epsilon^{d-4}} + \dots + C_2\frac{L^2}{\epsilon^2} + C_0 \log\frac{L}{\epsilon} \quad (2)$$

In odd dimensions

$$S_A = C_{d-2}\frac{L^{d-2}}{\epsilon^{d-2}} + C_{d-4}\frac{L^{d-4}}{\epsilon^{d-4}} + \dots + C_1\frac{L}{\epsilon} + (-1)^{\frac{d-1}{2}} F \quad (3)$$

These cases show the EE have area law in leading divergent term.

Main motivation

- In string theory, EE should be finite and we expect (in 10D)

$$S_A = s\frac{V_8}{\sqrt{\alpha'}^8} + \ldots \tag{4}$$

Replica to calculate EE in QFT

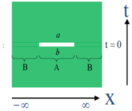
- One question: How to calculate EE in quantum system?
- A basic method of calculating EE in QFTs is so called the replica method.

$$S_A = -\frac{\partial \log \mathrm{Tr}(\rho_A)^n}{\partial n}|_{n=1} = \lim_{n\to 1} S_A^n$$

- The relation provides a practical way to compute EE in field theory, although it is difficult.
- In this talk, we would like to apply replica trick in string theory.

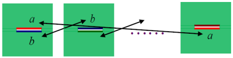
Replica vs Orbifold construction

- The density matrix is

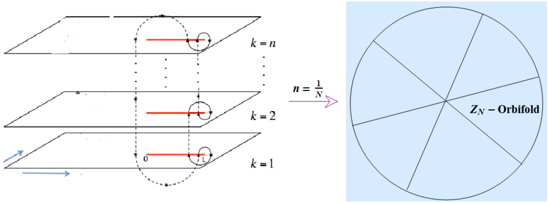


- Do the n copies of density matrix

$$\mathrm{Tr}\left(\rho_A\right)^n =$$



- In 2D real space, do analytical continuation $n = 1/N$ to obtain the orbifolds $C/\mathbb{Z}_N$



For vacuum state,

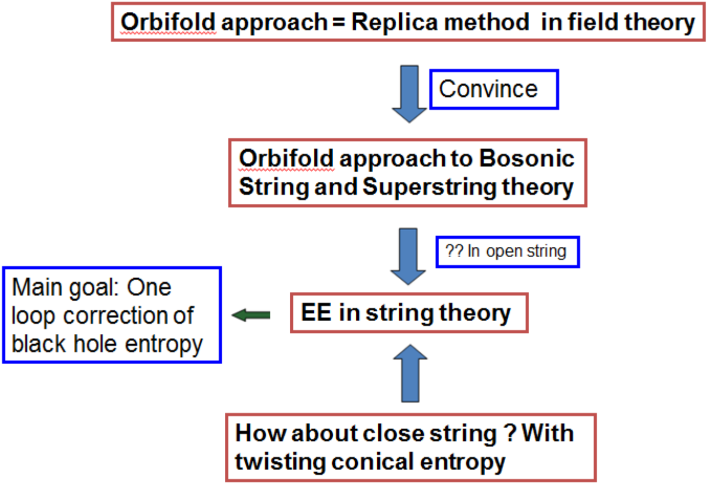
Entanglement entropy

$$S_A = -(\partial_n - 1)\log Z_n\big|_{n=1}$$

All we need to know is the partition function Z_n on the n-fold cover $\mathcal{M}_n$!

In this talk, we would like to apply replica trick in string theory.

Structure of this talk



New proposal for EE in field theory

Our Setup in Free Field Theory

- A free field theory on $M = R^D$, whose coordinate denoted by $(x_0, x_1, ..., x_{D-1})$. The subsystem A is $x_1 > 0$.
- Combining (x_0, x_1) into a complex plane C and the n-th Rényi entanglement entropy $S_A^{(n)}$,

$$S_A^{(n)} = \frac{1}{1-n}\left[Z^f(C/\mathbb{Z}_N \times R^{D-2}) - \frac{1}{N} Z^f(C \times R^{D-2})\right]\Bigg|_{N=\frac{1}{n}}. \quad (5)$$

The $\mathbb{Z}_N$ orbifold action g is given by

$$g : (X, \bar{X}) \to (e^{\frac{2\pi i}{N}} X, e^{\frac{-2\pi i}{N}} \bar{X}). \quad (6)$$

EE for Free Scalar

- Firstly, we take free massive scalar field theory as an example.
- The partition function of a spinless particle on the flat space

$$
\begin{aligned}
Z^f(R^D) &= \int_{\epsilon^2}^{\infty} \frac{ds}{2s} \mathrm{Tr}\, e^{-s(\hat{k}^2+m^2)} \qquad (7) \\
&= \frac{V_D}{(2\pi)^D} \int_{\epsilon^2}^{\infty} \frac{ds}{2s} \int d^D k \, e^{-s(k^2+m^2)} \\
&= V_D \int_{\epsilon^2}^{\infty} \frac{ds}{2s} (4\pi s)^{-\frac{D}{2}} e^{-sm^2}, \qquad (8)
\end{aligned}
$$

the parameter s is the Schwinger parameter which is a moduli in the first quantization approach.

EE for Free Scalar

- The partition function of a spinless particle on $\mathbb{Z}_N$ orbifold is

$$\begin{aligned} Z^f(C/\mathbb{Z}_N \times R^{D-2}) &= \int_{\epsilon^2}^{\infty} \frac{ds}{2s} \mathrm{Tr} \frac{1}{N} \sum_{j=0}^{N-1} g^j \, e^{-s(\hat{k}^2+m^2)} \\ &= \int_{\epsilon^2}^{\infty} \frac{ds}{2s} \int d^D k \, \frac{1}{N} \sum_{j=0}^{N-1} \left\langle \vec{k} \right| g^j \left| \vec{k} \right\rangle e^{-s(k^2+m^2)} \end{aligned} \quad (9)$$

  Here g is the generator of $\mathbb{Z}_N$. Due to twist boundary condition (BC), one should project out some modes related to the BC.

- Finally, we can get the REE from (5);

$$S_A^{(n)} = \frac{(n+1)\pi V_{D-2}}{6n} \int_{\epsilon^2}^{\infty} \frac{ds}{(4\pi s)^{\frac{D}{2}}} \, e^{-m^2 s}. \quad (10)$$

  Especially the coefficient V_{D-2} shows the area law of REE.

EE for Higher Spin Fields

- Generalize to all Higher spin Fields and Fermion fields with considering proper spin structure.
- Complicated details ...
- The generic expression of heat kernel expression for spin field [D. V. Fursaev and G. Miele, 96]

$$\log Z^{(j)} = (-1)^F \int_{\epsilon^2}^{\infty} e^{-m^2 s} \cdot (4\pi s)^{-D/2} \left(A_0^{(j)} + s A_1^{(j)} + \cdots\right) \int_{R^{D-2}}, \tag{11}$$

where $\int_{R^{D-2}}$ denotes the volume of R^{D-2}.

-

$$A_1^{(j)} = Q^{(j)} + 4\pi c_1^{(j)}(N-1) + O\left((N-1)^2\right), \tag{12}$$

where the constant $Q^{(j)}$ denotes a singular contribution, which is non-zero only for $j \geq 3/2$.

Compare with Heat Kernel

- We have checked conical entropy of spin $0, 1/2, 1, 3/2, 2$ which is the same as heat kernel method. In field theory side, there is no known results about beyond spin 2.
- The conical entropy (BH) (EE+surface($Q^{(j)}$)) is not protected to be positive with containing the surface term contributions in spin $1, 2, \ldots$ particles.

Conical Entropy in String Theory

The additional motivation

- The other main motivation is to understand BH entropy in terms of EE.
- Rindler spacetime can approximates the near horizon geometry of a large black hole. EE in field theory

$$S_A = s\frac{A}{\epsilon} + \dots \tag{13}$$

['t Hooft,85]

The additional motivation

- Three difficulties for EE $=$ BH Entropy
  1. EE in field theory is divergent V.S. BH entropy is finite due to Hawking formula.
  2. The EE has no classsical contribution and starts at one loop V.S. the BH entropy is inversely proportional to the coupling constant.
  3. The EE depends on couplings of various particles in the theory V.S. BH entropy does not.
- Let us move to String theory. Probably...

Conical Entropy in Some String theories

- For Bosonic string, there is no well defined EE due to IR divergent (Tachyon presence).
- For Open Superstring, the ToT conical entropy can be a summation of conical entropy for all higher spin fields(area law divergences).
- The divergence shows that the backreaction of open string sectors to closed string sectors is very important.
- To treat this backreaction properly motivates us to study the conical entropy in closed string.

(Twisted) Conical Entropy in Closed Superstring-1

- The sphere amplitude are expected to lead to the Bekenstein-Hawking formula.

$$S_{BH} = \frac{A}{4\pi G} \tag{14}$$

- We will focus on the torus amplitude corresponding to the leading quantum correction.
- We focus on type II string theory on the flat space $M = R^{10}$, whose coordinate denoted by $(x_0, x_1, \cdots, x_9)$. We define the subsystem A to be $x_1 > 0$.

(Twisted) Conical Entropy

- Then combining (x_0, x_1) into a complex plane C, we can introduce the n-th REE $S_A^{(n)}$ of string theory,

$$S_A^{(n)} = \frac{1}{1-n}\left[Z_{closed}(C/\mathbb{Z}_N \times R^8) - \frac{1}{N}Z_{closed}(C \times R^8)\right]\Big|_{N=\frac{1}{n}}, \tag{15}$$

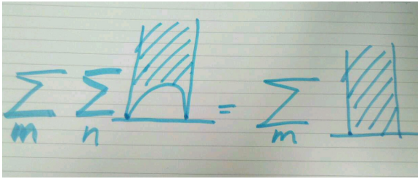
where $C/\mathbb{Z}_N$ is the standard $\mathbb{Z}_N$ orbifold in type II string. The $\mathbb{Z}_N$ orbifold action g is given by

$$g : (X, \bar{X}) \to (e^{\frac{2\pi ik}{N}}X, e^{\frac{-2\pi ik}{N}}\bar{X}), \tag{16}$$

where k is a positive integer fixed below.

Conical Entropy in string theory

- The partition function of type II string on $C/\mathbb{Z}_N \times R^8$ is...



Double sum is hard to treat. The alternative way?

- More precisely, $C/\mathbb{Z}_N \times R^8$ is

$$Z_{closed}(C/\mathbb{Z}_N \times R^8)$$
$$= V_8 \int_F \frac{d\tau^2}{4\tau_2} \cdot (4\pi^2\alpha'\tau_2)^{-4} \cdot \sum_{l,m=0}^{N-1} \frac{|\theta_1(\nu_{lm}/2|\tau)|^8}{N|\eta(\tau)|^{18}|\theta_1(\nu_{lm}|\tau)|^2} \quad (17)$$

where $\nu_{lm} = \frac{k(l-m\tau)}{N}$.

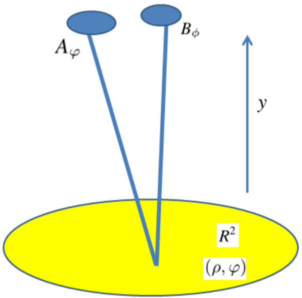
- In string theory on Melvin background, the folding trick can simplify this issue with T duality.

What is Melvin background

- Do slightly deformation or modification (Introducing Melvin background). The follow approach can lead us to preliminary results and heuristic understanding on EE.
- The target spaces of models have the structure of Kaluza-Klein (KK) theory with the topology $M_3 \times R^{1,6}$. M_3 is given by S^1 fibration over R^2.

What is Melvin background

- We just used following figure to show the structure



Here non-trivial two Kaluza-Klein (K.K.) gauge fields A_φ and B_ϕ originate from K.K. reduction of metric $G_{\varphi y}$ and B-field $B_{\varphi y}$, respectively.

What is Melvin background

- The explicit metric and other NSNS fields before the K.K. reduction are

$$
\begin{aligned}
ds^2 &= d\rho^2 + \frac{\rho^2}{(1+\beta^2\rho^2)(1+q^2\rho^2)} d\varphi^2 + \frac{1+q^2\rho^2}{1+\beta^2\rho^2}(dy + A_\varphi d\varphi)^2, \\
A_\varphi &= \frac{q\rho^2}{1+q^2\rho^2}, \quad B_{\varphi y} \equiv B_\varphi = -\frac{\beta\rho^2}{1+\beta^2\rho^2}, \quad e^{2(\phi-\phi_0)} = \frac{1}{1+\beta^2\rho^2}, \qquad (18)
\end{aligned}
$$

- q, β are proportional to the strength of two gauge fields $A_\varphi,\ B_\varphi$ as well as ϕ_0 is the constant value of the dilaton ϕ at $\rho = 0$.
- Here we have neglected the trivial flat part $\mathbf{R}^{1,6}$.

(Twisted) Conical Entropy

- The Melvin backgrounds is close to a $\mathbb{Z}_N$ orbifold (or called a twisted circle):

$$\text{Melvin background} : (C \times S^1)/\mathbb{Z}_N \times R^7, \qquad (19)$$

where the radius of the circle S^1 before the $\mathbb{Z}_N$ orbifold is defined to be NR.

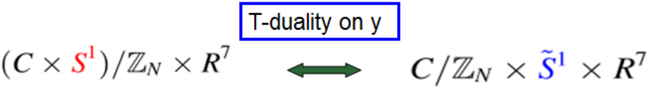
- The $\mathbb{Z}_N$ orbifold action g is defined by

$$g : (X, \bar{X}, y) \to (e^{\frac{2\pi i k}{N}} X, e^{\frac{-2\pi i k}{N}} \bar{X}, y + 2\pi R), \qquad (20)$$

where k is even and N is odd integer.

(Twisted) Conical Entropy

- (19) can be reduce to original orbifold $C/\mathbb{Z}_N \times \tilde{S}^1 \times R^7$ if we take $R \to 0$ using T-duality. The T-dualized radius $\tilde{S}$ is $R_{orb} = \frac{\alpha'}{NR}$.



(Twisted) Conical Entropy

- The twisted conical entropy as follows:

$$\tilde{S}_A^{(n)} = \frac{1}{1-n}\left[Z_{closed}\left[(C\times S^1)/\mathbb{Z}_N\times R^7\right]-\frac{1}{N}Z_{closed}\left[C\times S^1\times R^7\right]\right]\Big|_{n=1/N}. \quad (21)$$

- The von-Neumann entropy can be computed as

$$\begin{aligned}\tilde{S}_A &\equiv \tilde{S}_A^{(1)} = Z_{closed}\left[C\times S^1\times R^7\right]\\ &+ \frac{\partial}{\partial N}Z_{closed}\left[(C\times S^1)/\mathbb{Z}_N\times R^7\right]\Big|_{N=1}, \end{aligned}\quad (22)$$

- We expect $S_A\left[C\times R^8\right]=\lim_{R_{orb}\to\infty}\tilde{S}_A\left[C\times S^1\times R^7\right]$.

(Twisted) Conical Entropy

- The partition function of Melvin model [J.G.Russo, A.A. Tseytlin,95][T.Takayanagi and T. Uesugi][...] is given by

$$
\begin{aligned}
& Z_{closed}\left[(C \times S^1)/\mathbb{Z}_N \times R^7\right] \\
= \quad & Z_0 \cdot \int_F \frac{d\tau^2}{\tau_2^5} \sum_{w',w=-\infty}^{\infty} e^{-\frac{\pi R^2}{\alpha' \tau_2}|w-w'\tau|^2} \\
\cdot \quad & \frac{|\theta_1((w-w'\tau)/N|\tau)|^8}{|\eta(\tau)|^{18}|\theta_1(2(w-w'\tau)/N|\tau)|^2},
\end{aligned}
\tag{23}
$$

where $Z_0 = \frac{V_7 R}{4(2\pi)^7 \alpha'^4}$. The region F represents the standard fundamental region of the torus moduli space.

(Twisted) Conical Entropy

- Folding trick enables us to rewrite it as the integral over the strip S defined by $-1/2 < \tau_1 < 1/2$ and $\tau_2 > 0$, with a single sum:

$$Z_{closed}\left[(C \times S^1)/\mathbb{Z}_N \times R^7\right] = Z_0 \int_S \frac{d\tau^2}{\tau_2^5} \sum_{w=-\infty}^{\infty} e^{-\frac{\pi R^2}{\alpha' \tau_2} w^2} \cdot \frac{|\theta_1(w/N|\tau)|^8}{|\eta(\tau)|^{18} \cdot |\theta_1(2w/N|\tau)|^2}, \quad (24)$$

where the integral region S denotes the strip defined by $-1/2 < \tau_1 < 1/2$ and $\tau_2 > 0$.

- $\omega = N\alpha + \beta$ and the α runs all integers from $-\infty$ to ∞, while β takes $0, 1, 2, ..., N-1$.

(Twisted) Conical Entropy

- After we did the Poisson resummation:

$$\sum_{\gamma\in\mathbb{Z}} \exp(-\pi a\gamma^2 + 2\pi i b\gamma) = \frac{1}{\sqrt{a}} \sum_{\alpha'\in\mathbb{Z}} \exp(-\frac{\pi(\alpha-b)^2}{a}), \tag{25}$$

- we find

$$Z_{closed}\left[(C\times S^1)/\mathbb{Z}_N \times R^7\right] \tag{26}$$

$$= Z_0 \int_S \frac{d\tau^2}{\tau_2^5} \frac{\sqrt{\alpha'\tau_2}}{NR} \sum_{\gamma\in\mathbb{Z}} \sum_{\beta=0}^{N-1} e^{-\frac{\pi\alpha'\tau_2}{R^2N^2}\gamma^2} \cdot e^{2\pi i\frac{\beta\gamma}{N}} \frac{|\theta_1(\beta/N|\tau)|^8}{|\eta(\tau)|^{18}\cdot|\theta_1(2\beta/N|\tau)|^2}. \tag{27}$$

In the following, we will study this function in IR $\tau\to 0$ and UV region $\tau\to\infty$.

Short Summary for twist Conical Entropy

- The moduli integral of the twisted conical entropy $\tilde{S}_A$ does converge both in the IR ($\tau \to 0$) and UV ($\tau \to \infty$) region.
- In bosonic string Melvin backgrounds, the twisted conical entropy will get divergent due to the tachyon.

Short Summary for twist Conical Entropy

- In the large T-dual radius limit $R_{orb} \to \infty$, the twisted conical entropy behaves as follows:

$$\tilde{S}_A(R_{orb}) \simeq \tilde{s} \cdot \frac{V_7}{\alpha'^{7/2}}. \tag{28}$$

Which is finite. But there is no area law here (One loop quantum correction to BH is 0).

- In superstring, this UV divergence can be removed owing to the string scale cutoff ($\sqrt{\alpha'} = l_s$).

Conical Entropy vs EE

- By definition, Conical Entropy = EE(>0) + Surface term contribution.
- Due to SUSY, there is nature cancelation for Conical Entropy.[L. Susskind, J. Uglum, 94]
- In terms of BH entropy for the Rindler horizon, Conical entropy from string theory one-loop amplitudes corresponds to the leading quantum correction.[L. Susskind, J. Uglum, 94]
- In twist EE, there is no area law term and it means no one loop quantum correction to BH entropy.

Summary and comments

1. Reproducing Conical entropy for spin 0,1/2,1,3/2 and 2 and offer for any higher spins fields. (Nice Check and apply this logic to String theory.)
2. The conical entropy for open string turns out to be divergent even in superstring. The back reactions of open strings to the closed string sector. (Future work).
3. The twisted conical entropy $\tilde{S}_A$ is confirmed to be finite.
4. The twisted conical entropy $\tilde{S}_A$ does not have any contributions related to the boundary ∂A. Indirect signal to support the quantum corrections to conical entropy S_A in type II closed superstring is vanishing. (Future Problem).
5. Higher genus string amplitude. (Future Problem)

Thanks for your attention!