Precision limit in quantum state tomography

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Outline

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Introduction

- Quantum state estimation is a procedure for inferring the state of a quantum system from generalized measurements, known as positive-operator-valued measures (POVMs).
- Quantum state estimation is a primitive of quantum computation, quantum cryptography, and many other quantum information processing tasks.
- It usually requires many copies of the unknown true states to reach sufficient accuracy.
- A main goal of current research on quantum state estimation is to reconstruct generic unknown quantum states as efficient as possible.

Quantum precision limit: Foundational perspective

- The precision limit in quantum state estimation is of great interest not only to practical applications but also to foundational studies.
- Little is known about this subject in the multiparameter setting even theoretically.
- The difficulty is closely related to the existence of incompatible observables, which underly many nonclassical phenomena, such as uncertainty relations, wave-particle dual behavior, Bell-inequality violation, contextuality, and superdense coding.
- Advances in understanding the quantum precision limit and these foundational problems are mutually beneficial.

Informationally complete measurements

- A POVM is composed of a set of outcomes represented mathematically by positive operators Π_ξ satisfying $\sum_\xi \Pi_\xi = 1$.
- Given a state ρ, the probability of obtaining outcome Π_ξ is given by the Born rule $p_\xi = \mathrm{Tr}(\Pi_\xi \rho)$.
- A POVM is informationally complete (IC) if we can reconstruct any state according to the statistics of measurement results, that is the set of probabilities p_ξ.

State reconstruction

- Linear tomography: $\hat{\rho} = \sum_\xi f_\xi \Theta_\xi$ given reconstruction operators Θ_ξ and frequencies $f_\xi = n_\xi / N$.
- Maximum-likelihood estimation: the estimator maximizes the likelihood functional

$$\mathcal{L}(\rho) = \prod_\xi p_\xi^{n_\xi}.$$

- Bayesian mean estimation.
- Hedged maximum-Likelihood estimation.
- State estimation based on maximum-entropy principle, compressed sensing...

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Figures of merit

- Hilbert-Schmidt (HS) distance

$$\|\rho-\sigma\|_{\mathrm{HS}} = \sqrt{\operatorname{tr}(\rho-\sigma)^2}.$$

- Trace distance

$$\|\rho-\sigma\|_{\mathrm{tr}} = \frac{1}{2}\operatorname{tr}|\rho-\sigma|.$$

- Fidelity and Bures distance

$$F(\rho,\sigma) = \left(\operatorname{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right)^2 = \left(\operatorname{tr}|\rho^{1/2}\sigma^{1/2}|\right)^2,$$
$$D_{\mathrm{B}}^2(\rho,\sigma) = 2-2\sqrt{F(\rho,\sigma)}.$$

Measurements on composite systems

- A measurement on a composite system is separable if each outcome can be written as a convex combination of tensor products of positive operators or, equivalently, if each outcome corresponds to a separable state.
- A measurement is entangled if it is not separable.

Questions

1. What is the precision limit of quantum state estimation?
2. By how much can the precision be increased with entangled measurements compared with separable measurements?

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Fisher information

- Consider a family of probability distributions $p(\xi|\theta)$ parameterized by θ. Given an outcome ξ, the function $p(\xi|\theta)$ of θ is called a likelihood function.
- The score is defined as the partial derivative of the log-likelihood function with respect to θ.
- The score has a vanishing first moment; its second moment is known as the Fisher information [Fis22],

$$I(\theta) = \sum_{\xi} p(\xi|\theta)\left(\frac{\partial \ln p(\xi|\theta)}{\partial\theta}\right)^2 = \sum_{\xi} \frac{1}{p(\xi|\theta)}\left(\frac{\partial p(\xi|\theta)}{\partial\theta}\right)^2.$$

- Multi-parameter setting:

$$I_{jk}(\theta) = \mathrm{E}\left[\left(\frac{\partial \ln p(\xi|\theta)}{\partial\theta_j}\right)\left(\frac{\partial \ln p(\xi|\theta)}{\partial\theta_k}\right)\right].$$

Cramér-Rao bound

- An estimator $\hat{\theta}(\xi)$ of the parameter θ is unbiased if its expectation value is equal to the true parameter,

$$\sum_{\xi} p(\xi|\theta)(\hat{\theta}(\xi) - \theta) = 0.$$

- Cramér-Rao (CR) bound: the mean square error (MSE) of any unbiased estimator is bounded from below by the inverse of the Fisher information [Cra46, Rao45],

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)}.$$

- Multi-parameter setting:

$$C(\theta) \geq I^{-1}(\theta), \quad \mathrm{tr}\{W(\theta)C(\theta)\} \geq \mathrm{tr}\{W(\theta)I^{-1}(\theta)\},$$

where C is the MSE matrix, and W is a weighting matrix.

- In quantum state estimation, we are interested in the parameters that characterize the state $\rho(\theta)$ of a quantum system.
- Given a measurement Π with outcomes Π_ξ, the probability of obtaining outcome ξ is $p(\xi|\theta) = \mathrm{tr}\{\rho(\theta)\Pi_\xi\}$. The Fisher information reads

$$I_{jk}(\Pi, \theta) = \sum_\xi \frac{1}{p(\xi|\theta)} \mathrm{tr}\left\{\frac{\partial\rho(\theta)}{\partial\theta_j}\Pi_\xi\right\} \mathrm{tr}\left\{\frac{\partial\rho(\theta)}{\partial\theta_k}\Pi_\xi\right\}.$$

- The inverse Fisher information matrix sets a lower bound for the MSE matrix of any unbiased estimator. However, the bound depends on the specific measurement.

Quantum Fisher information

- Let $\rho'(\theta) = \mathrm{d}\rho(\theta)/\mathrm{d}\theta$. A Hermitian operator $L(\theta)$ satisfying the equation
$$\rho'(\theta) = \frac{1}{2}[\rho(\theta)L(\theta) + L(\theta)\rho(\theta)]$$
is called the symmetric logarithmic derivative (SLD) of $\rho(\theta)$ with respect to θ [Hel76, Hol82].
- The SLD satisfies $\operatorname{tr}\{\rho(\theta)L(\theta)\} = 0$ and
$$\operatorname{tr}\{\rho'(\theta)A\} = \Re\operatorname{tr}\{\rho(\theta)L(\theta)A\} = \Re\operatorname{tr}\{\rho(\theta)AL(\theta)\}$$
for any Hermitian operator A.
- SLD quantum Fisher information [Hel76, Hol82]
$$J(\theta) = \operatorname{tr}\{\rho(\theta)L(\theta)^2\}.$$

Quantum Cramér-Rao bound

- $$\begin{aligned} I(\theta) &= \sum_\xi \frac{[\mathrm{tr}(\rho'\Pi_\xi)]^2}{\mathrm{tr}(\rho\Pi_\xi)} \leq \sum_\xi \frac{|\mathrm{tr}(\rho\Pi_\xi L)|^2}{\mathrm{tr}(\rho\Pi_\xi)} \\ &= \sum_\xi \frac{\left|\mathrm{tr}\left\{(\Pi_\xi^{1/2}\rho^{1/2})^\dagger \Pi_\xi^{1/2} L \rho^{1/2}\right\}\right|^2}{\mathrm{tr}(\rho\Pi_\xi)} \\ &\leq \sum_\xi \mathrm{tr}\{\rho L \Pi_\xi L\} = \mathrm{tr}(\rho L^2) = J(\theta), \end{aligned}$$
- The two inequalities can be saturated simultaneously by measuring the observable $L(\theta)$.
- In the multi-parameter setting,
  $$J_{jk} = J_{kj} = \frac{1}{2}\mathrm{tr}\left\{\rho(L_j L_k + L_k L_j)\right\}.$$
  The inequality $I \leq J$ generally cannot be saturated unless the L_j commute with each other.

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- The SLD bound for the scaled mean square Hilbert-Schmidt distance (MSH),

$$\mathcal{E}_{\mathrm{SH}}^{\mathrm{SLD}}(\rho) = d - \mathrm{tr}(\rho^2).$$

- The SLD bound for the scaled mean square Bures distance (MSB),

$$\mathcal{E}_{\mathrm{SB}}^{\mathrm{SLD}}(\rho) = \frac{d^2 - 1}{4}.$$

Precision of minimal tomography

- The scaled MSH achievable with linear or minimal tomography [Sco06],

$$\mathcal{E}_{\mathrm{SH}}(\rho) = d^2 + d - 1 - \mathrm{tr}(\rho^2).$$

 The minimum can be achieved by SIC measurements or any measurement constructed out of a 2-design.

- The scaled mean trace distance:

$$\mathcal{E}_{\mathrm{tr}}(\rho) \approx \frac{4}{3\pi}\sqrt{d\mathcal{E}_{\mathrm{SH}}(\rho)} \sim \frac{4}{3\pi} d^{3/2}.$$

Outline

Gill-Massar inequality

Gill-Massar trace (GMT) [GM00] $t(\theta) := \mathrm{tr}\{J^{-1}(\theta)I(\theta)\}$.

Theorem (Gill-Massar, 2000)

The inequality

$$\mathrm{tr}\{J^{-1}(\theta)I(\theta)\} \leq N(d-1).$$

holds for any separable measurement on N copies of the true state. The bound is saturated for any rank-one measurement when the number of parameters is equal to $d^2 - 1$.

This theorem succinctly summarizes the information trade-off in quantum state estimation in the multi-parameter setting, which implies that it is generally impossible to construct a measurement that is optimal for all parameters.

Gill-Massar bound

- The GM bound for the weighted mean square error (WMSE),

$$\mathcal{E}_W^{\mathrm{GM}} = \frac{\left(\mathrm{tr}\sqrt{W^{1/2}J^{-1}W^{1/2}}\right)^2}{d-1} = \frac{\left(\mathrm{tr}\sqrt{J^{-1/2}WJ^{-1/2}}\right)^2}{d-1}.$$

- The GM bounds for the MSB and MSH

$$\mathcal{E}_{\mathrm{SB}}^{\mathrm{GM}}(\rho) = \frac{1}{4}(d+1)^2(d-1),$$

$$\mathcal{E}_{\mathrm{SH}}^{\mathrm{GM}}(\rho) = \frac{1}{d-1}\left(\sum_{j\neq k=0}^{d-1}\sqrt{\frac{\lambda_j+\lambda_k}{2}} + \mathrm{tr}\sqrt{\Lambda}\right)^2,$$

  where the λ_j are the eigenvalues of ρ, and Λ is the $d \times d$ matrix with $\Lambda_{jk} = \lambda_j\delta_{jk} - \lambda_j\lambda_k$.

- In the case of a qubit, the GM bound can be saturated; little is known in general.

Approximate joint measurement of complementary observables

- The impossibility of measuring simultaneously complementary observables, say σ_x and σ_z, is closely related to wave-particle dual behavior.
- Observables $\mathbf{A} = \{(1 \pm \eta_x \sigma_x)/2\}$ and $\mathbf{B} = \{(1 \pm \eta_z \sigma_z)/2\}$ are compatible if and only if (Busch86)

$$\eta_x^2 + \eta_z^2 \leq 1.$$

This inequality follows from general compatibility criteria inspired by quantum estimation theory and data-processing inequalities.

H. Zhu*, *Information complementarity: A new paradigm for decoding quantum incompatibility*, Sci. Rep. **5**, 14317 (2015).
H. Zhu*, M. Hayashi, L. Chen, *Universal Steering Criteria*, Phys. Rev. Lett. **116**, 070403 (2016).

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Precision limit of qubit state estimation with adaptive and nonadaptive measurements

Scaled MSE achievable with SIC, MUB, and adaptive measurements,

$$\begin{aligned}\mathcal{E}^{\mathrm{SIC}}(s) &= (9 - s^2),\\ \mathcal{E}^{\mathrm{MUB}}(s) &= 3(3 - s^2),\\ \mathcal{E}^{\mathrm{Adaptive}}(s) &= (2 + \sqrt{1 - s^2})^2.\end{aligned}$$

Scaled MSB achievable with SIC, MUB, and adaptive measurements,

$$\begin{aligned}\overline{\mathcal{E}^{\mathrm{SIC}}_{\mathrm{SB}}(s)} &= \frac{9}{4} + \frac{s^2}{2(1 - s^2)},\\ \overline{\mathcal{E}^{\mathrm{MUB}}_{\mathrm{SB}}(s)} &= \frac{9}{4} + \frac{3s^4}{10(1 - s^2)},\\ \mathcal{E}^{\mathrm{Adaptive}}_{\mathrm{SB}}(s) &= \frac{9}{4}.\end{aligned}$$

Qubit state estimation with two-step adaptive measurements

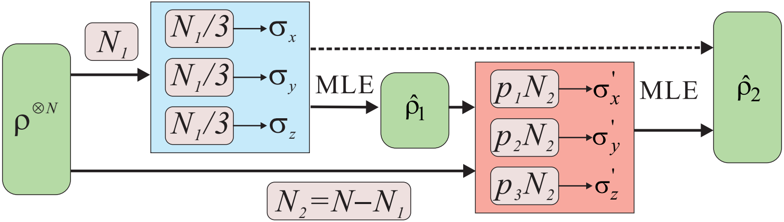

Figure: (color online) The observables σ'_x, σ'_y, σ'_z and the probabilities p_1, p_2, p_3 depend on both the estimator $\hat{\rho}_1$ obtained in the first step and the figure of merit. In the large-N limit, it suffices to use the measurement statistics of step 2 to construct the second MLE. In practice, it is preferable to employ the measurement statistics of both steps.

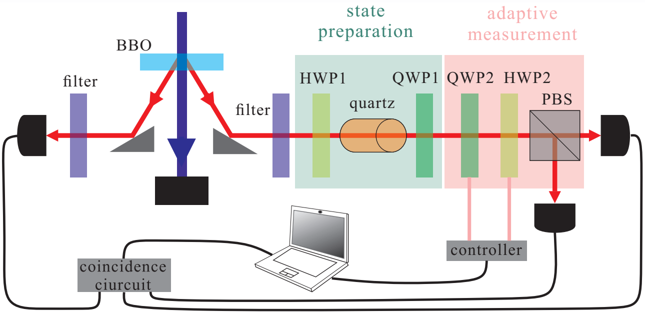

Figure: The experimental setup of Zhibo and Guoyong at USTC. A pair of horizontally polarized photons are generated via pumping a barium borate (BBO) crystal. One is detected as a trigger and the other is sent through a half-wave plate (HWP), a quarter-wave plate (QWP), and a 400λ-quartz crystal in between, which serve as the state preparation module (green). The adaptive measurement module (pink) is composed of QWP2, HWP2, a polarizing beam splitter (PBS), and two photon detectors.

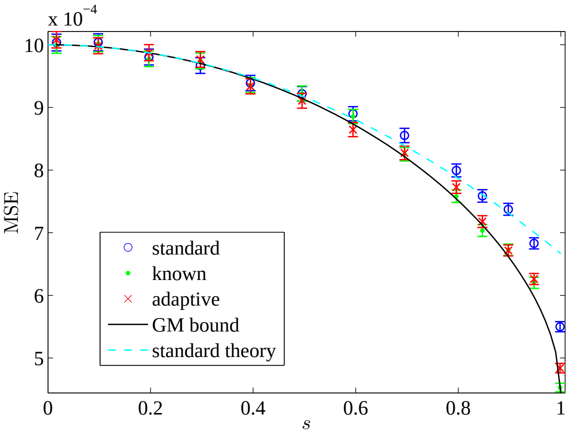

Figure: Precision limit with respect to the MSE. Experimental results of standard, adaptive, and known-state tomography are shown together with the theoretical MSE of the standard tomography and the GM bound. Here s is the length of the Bloch vector.

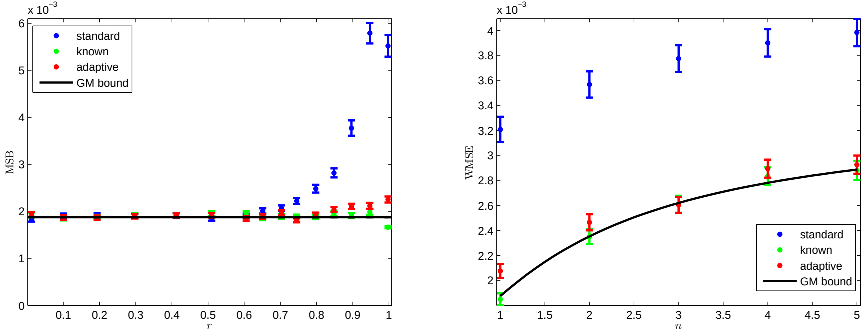

Figure: Left: The MSB of standard (blue), adaptive (red), known-state (green) measurements together with the GM bound (black). Right: The WMSEs with respect to a family of monotone Riemannian metrics (including the Bures metric $n = 1$ and the quantum Chernoff metric $n = 2$) for a state with $r = 0.9$.

Z. Hou, **H. Zhu***, G.-Y. Xiang, C.-F. Li, and G.-C. Guo, *Achieving quantum precision limit in adaptive qubit state tomography*, npj Quantum Information, **2**, 16001 (2016).

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Precision limit in asymptotic state estimation

When arbitrary collective measurements are allowed, the precision limit is determined by the quantum Cramér-Rao bound based on the right logarithmic derivative (RLD).
The RLD bound for the scaled MSE reads

$$\mathcal{E}^{\mathrm{RLD}} = d - \mathrm{tr}(\rho^2) + \sum_{k>j=1}^{d} |\lambda_j - \lambda_k|,$$

The minimum $d - 1/d$ is attained when ρ is the completely mixed state, and the maximum $2(d-1)$ when ρ is pure.
The RLD bound for the scaled MSB reads

$$\mathcal{E}_{\mathrm{SB}}^{\mathrm{RLD}} = \frac{d^2-1}{4} + \frac{1}{2}\sum_{k>j=1}^{d} \frac{|\lambda_j - \lambda_k|}{\lambda_j + \lambda_k}.$$

The minimum $(d-1)(d+1)/4$ is attained at the completely mixed state, and the supremum $(d-1)(2d+1)/4$ in the limit $\lambda_j/\lambda_{j-1} \to 0$ for $j = 2, 3, \ldots, d$.

The maximal scaled GMT

$$t^{\mathrm{RLD}} = d - 1 + \sum_{k>j=1}^{d} \frac{\lambda_j + \lambda_k}{\max(\lambda_j, \lambda_k)}.$$

The maximum $d^2 - 1$ is attained at the completely mixed state, and the infimum $(d-1)(d+2)/2$ in the limit $\lambda_j/\lambda_{j-1} \to 0$ for $j = 2, 3, \ldots, d$.

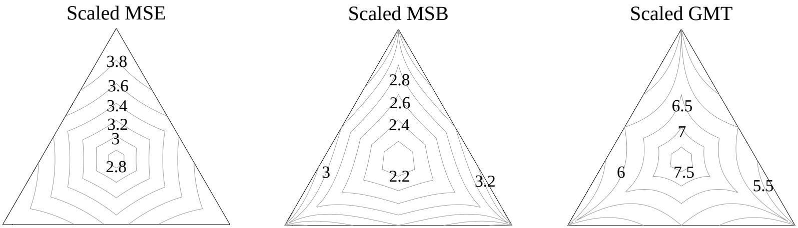

Figure: Contour plots of the asymptotic minimal scaled MSE, MSB, and maximal scaled GMT in the eigenvalue simplex for $d = 3$.

Optimal measurements on two copies of a qubit state

- POVM elements

$$|00\rangle\frac{1}{2}\langle 00|, \quad |11\rangle\frac{1}{2}\langle 11|, \quad |++\rangle\frac{1}{2}\langle ++|, \quad |--\rangle\frac{1}{2}\langle --|,$$
$$|\tilde{+}\tilde{+}\rangle\frac{1}{2}\langle\tilde{+}\tilde{+}|, \quad |\tilde{-}\tilde{-}\rangle\frac{1}{2}\langle\tilde{-}\tilde{-}|, \quad |\psi\rangle\langle\psi|.$$

  Here $|0\rangle, |1\rangle$ are eigenstates of σ_z; $|+\rangle, |-\rangle$ are eigenstates of σ_x; $|\tilde{+}\rangle, |\tilde{-}\rangle$ are eigenstates of σ_y; $|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$.

- Scaled Gill-Massar trace

$$\mathrm{tr}\{J^{-1}(s)I(s)\} = \frac{3}{2}$$

- Scaled mean square error and mean square Bures distance

$$\mathcal{E}(s) = 3 - s^2, \quad \mathcal{E}_{\mathrm{SB}}(s) = \frac{3}{2}.$$

No adaptive measurement is necessary.

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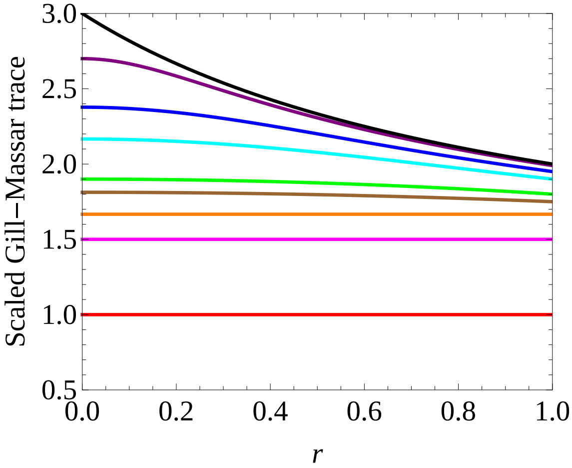

Figure: Maximal scaled GMT over all measurements on N copies of a qubit state for $N = 1, 2, 3, 4, 5, 10, 20, 100, \infty$ from bottom to top. The maximum is achieved for any coherent measurement. When $N = 1, 2, 3$, it is independent of r; otherwise, it decreases with r.

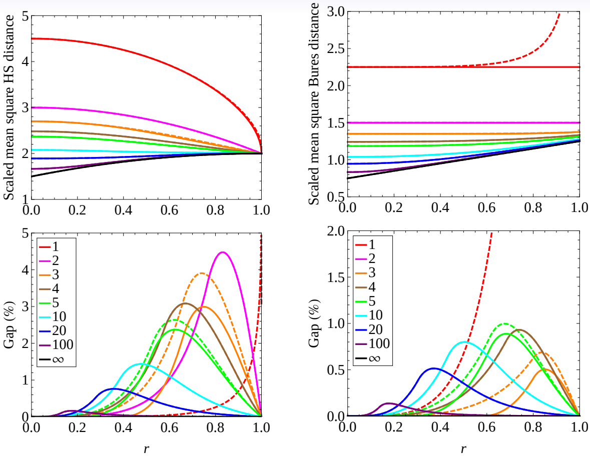

Figure: Scaled MSH (upper left) and scaled MSB (upper right) of the covariant coherent measurement (dashed) and the optimal coherent measurement (solid), respectively, on N copies of a qubit state for $N = 1, 2, 3, 4, 5, 10, 20, 100, \infty$ from top to bottom. The performances of the two kinds of measurements are identical for even N.

Outline

Summary

- We have presented an overview of the precision limits and optimal tomographic strategies under both separable measurements and entangled measurements.
- The distinctive features of each setting and the efficiency gaps between these settings were discussed in detail.
- Our study also highlighted the connection between quantum state estimation and several foundational issues, such as the complementarity principle, uncertainty relations, and quantum steering.

Current and future work

1. Determine the precision gap between optimal adaptive measurements and the Gill-Massar bound.
2. Propose tomographic protocols to achieve the precision limit of entangled measurements.
3. Propose reliable and efficient tomographic protocols capable of characterizing large quantum systems underlying quantum computation (more than 14 qubits).
4. Explore quantum metrology, quantum control, quantum sensing, and weak measurements.
5. Further explore the connection between quantum estimation theory and other foundational issues.

References

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Thank You!